A Premise to the Elliptic Curve Primality Test

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Note

This presentation is an overview of elliptic curve and its fundamental properties. This presentation serves as a meaningful transition to the morphism of curves, the degree of a morphism, Hasse's Theorem, and eventually Goldwasser-Kilian with its algorithm.

Presentation Overview

Affine Variety

Affine n-space

$$\mathbb{A}^n = \mathbb{A}^n(\bar{k}) = \{p = (x_1, \ldots, x_n) \mid x_i \in \bar{k}\}$$

Set of K-rational points in A^n

$$\mathbb{A}^n(k) = \{p = (x_1, \ldots, x_n) \in \mathbb{A}^n \mid x_i \in k\}$$

Affine Algebraic Set and Ideal

$$V_I = \{p \in \mathbb{A}^n \mid f(p) = 0 \text{ for all f} \in I\}$$
$$I(V) = \{f \in \bar{k}[x] \mid f(p) = 0 \text{ for all } p \in V\}.$$

1. $\bar{k}[x]$: $\bar{k}[x_1, \ldots, x_n]$

Affine Variety

An affine algebraic set V such that $I(V)$ is a prime ideal in $\bar{k}[x]$

Hilbert's Nullstellensatz for Affine Algebraic Sets

(a) For any algebraic set $X \subset \mathbb{A}^n$, we have

$$V(I(X)) = X.$$

(b) For any ideal $J \subset \bar{k}[x]$, we have

$$I(V(J)) = \sqrt{J}.$$

$$\{\text{algebraic sets in } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } k[x_1, \ldots, x_n]\}$$

Projective Variety

Projective n-space

$$\mathbb{P}^n(k) := \{[x_0, \ldots, x_n] \mid (x_0, \ldots, x_n) \in \mathbb{A}^{n+1}, \text{ not all } x_i = 0\}$$

- $[x_0 : \cdots : x_n]$ denotes an equivalence class under $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) \iff \exists \lambda \in \bar{k}^*$ such that $x_i = \lambda y_i \ \forall i$.

Homogeneous Polynomial

A polynomial $f \in k[x_0, \ldots, x_n]$ is homogeneous of degree d if

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n) \quad \text{for all } \lambda \in \bar{k}.$$

Homogeneous Ideal

$I \subset k[x_0, \ldots, x_n]$ is homogeneous if it is generated by homogeneous polynomials.

Projective Variety

Projective Algebraic Set and Ideal

$$V_I = \{p \in \mathbb{P}^n(k) \mid f(p) = 0 \text{ for all } f \in I\}, I \text{ homogeneous.}$$
$$I(V) = \{f \in \bar{k}[x] \mid f \text{ homogeneous and } f(p) = 0 \ \forall p \in V\}$$

Projective Variety

A projective algebraic set V such that $I(V)$ is a prime ideal in $k[x]$.

Smoothness

In affine space, let V be a variety, $p \in V$, and let $f_1, \ldots, f_m \in \bar{k}[x]$ be generators of $I(V)$.

- The Jacobian matrix at p is

$$J(p) = \left(\frac{\partial f_i}{\partial x_j}(p) \right)_{1 \le i \le m,\ 1 \le j \le n}.$$

- If $J(p)$ has rank $n - \dim(V)$, then V is nonsingular at p.
- If V is nonsingular at every point, then V is smooth everywhere.

In projective space, let $V \subset \mathbb{P}^n$ be a projective variety and $p \in V$. Choose an affine chart $\mathbb{A}^n \subset \mathbb{P}^n$ with $p \in \mathbb{A}^n$.

- V is nonsingular at p if $V \cap \mathbb{A}^n$ is nonsingular at p.
- This definition is independent of the chosen affine chart.

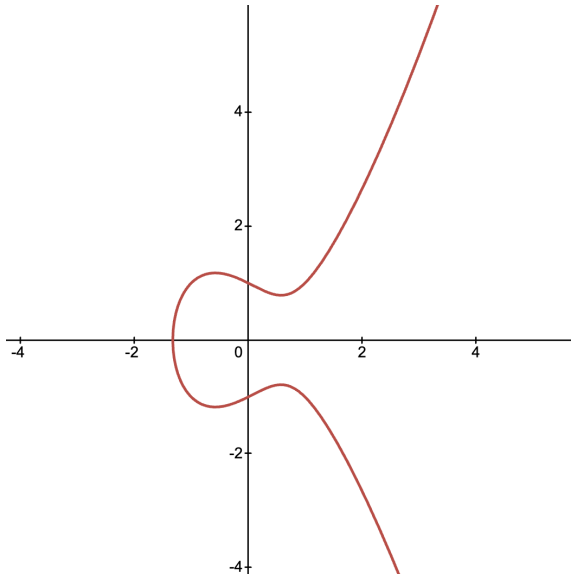
Definition of an Elliptic Curve (EC)

Elliptic Curve

An elliptic curve is a smooth projective curve of genus 1 with a distinguished point.

Example

$$y^2 = x^3 - x + 1$$



Projective Plane and Projective Plane Curves

Projective Plane

The set $\mathbb{P}^2(k)$ of all nonzero triples (x, y, z) in k^3 modulo the equivalence relation $(x, y, z) \sim (\lambda x, \lambda y, \lambda z), \quad \lambda \in k^\times$.

- Affine points: $(x : y : 1)$ Points at infinity: $(x : y : 0)$.

Plane Projective Curve

A homogeneous polynomial $f(x, y, z)$ with coefficients in k. The degree of C_f is the degree of $f(x, y, z)$.

K-rational points of C_f

For any field K containing k, the K-rational points of C_f form the set $C_f(K) = \{(x : y : z) \in \mathbb{P}^2(K) \mid f(x, y, z) = 0\}$.

- Smoothness is previously defined.

Weierstrass Equation

Weierstrass Equation

Let $A, B \in k$ with $4A^3 + 27B^2 \neq 0$, and assume $\mathrm{char}(k) \neq 2, 3$. The short Weierstrass equation stated as

$$y^2 = x^3 + Ax + B$$

defines a smooth projective curve of genus 1 over k with the rational point $(0 : 1 : 0)$.

- Up to isomorphism, every elliptic curve over k can be defined this way.
- $y^2z = x^3 + Axz^2 + Bz^3 \Rightarrow z = 0 \Rightarrow x = 0 \Rightarrow (0 : y : 0) \sim (0 : 1 : 0)$

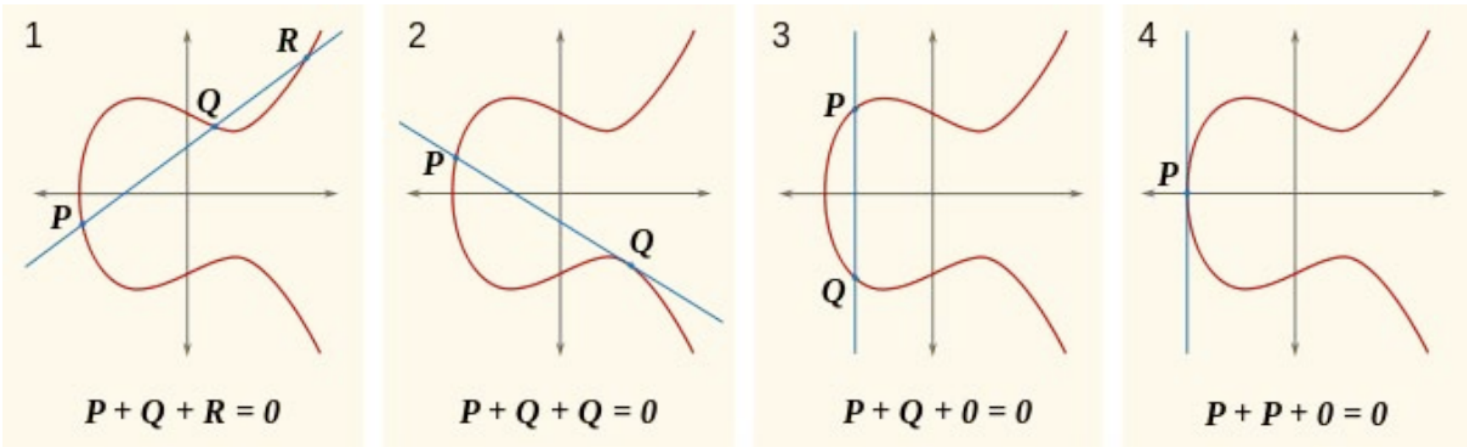
Notice

$4A^3 + 27B^2 = 0$ creates a singular point.

The Group Law

Three points on a line sum to zero.

- (P, Q, R)
- $P + Q = -R$, the reflection of R across the x-axis.



Properties

- Identity: The point $(0:1:0)$ at infinity is the identity element 0.
- Inverse: The inverse of $P = (x:y:z)$ is $-P = (x:-y:z)$.
- Commutativity: $P + Q = Q + P$.
- Associativity: $P + (Q + R) = (P + Q) + R$.
- Scalar Multiplication: $0P = 0$, $nP = P$

Notice

Commutativity is trivial, as the line passing through P and Q is the same as the line passing through Q and P. Associativity, on the other hand, is not so obvious.

The group law is algebraic and field independent. It works over $\mathbb{Q}$, $\mathbb{R}$, *and* $\mathbb{F}_p$.

References

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- A. Sutherland, *18.783: Elliptic Curves — Lecture Notes*, Spring 2021, MIT OpenCourseWare, Massachusetts Institute of Technology, https://ocw.mit.edu/courses/18-783-elliptic-curves-spring-2021/pages/lecture-notes-and-worksheets/.
- J. H. Silverman, *The Arithmetic of Elliptic Curves*, 2nd ed., Graduate Texts in Mathematics, vol. 106, Springer, 2009.